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Dense Subsets of $\mathscr{L}^1$-Solutions to Linear Elliptic Partial Differential Equations

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Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be an unbounded domain, and L_m be a homogeneous linear elliptic partial differential operator with constant coefficients. In this paper we show, among other things, that rapidly decreasing $\mathscr{L}^1$-solutions to L_m (in Ω) approximate all $\mathscr{L}^1$-solutions to L_m (in Ω), provided there exist real numbers $R_j \to \infty$, $\varepsilon \geqslant 0$, and a sequence $\{y^j\}$ such that $B(y^j, \varepsilon) \cap \Omega = \varnothing$ and

$$\frac{|\Lambda(y^j, R_j, \mathbb{R}^N \backslash \Omega)|}{R_j^N} > \varepsilon \qquad \forall j,$$

where $|\cdot|$ means the volume and

$$\Lambda(z, R, D) := \bigcup_{x \in B(z, R) \cap D} \left\{ z + t \frac{(x-z)}{|x-z|} ; t \leqslant 1 \right\},$$

for $z \in \mathbb{R}^N$, $R > 0$ and $D \subset \mathbb{R}^N$. For $m = 2$, we can replace the volume density by the capacity-density.

It appears that the problem is related to the characterization of largest sets on which a nonzero polynomial solution to L_m may vanish, along with its $(m-1)$-derivatives. We also study a similar approximation problem for polyanalytic functions in $\mathbb{C}$. © 2000 Academic Press

Key Words: polyanalytic functions; higher order elliptic pde; $\mathscr{L}^1$-approximation; dense subsets.

Contents

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NOTATION

C	a generic constant
χ_D	the characteristic function of the set D, $(D \subset \mathbb{R}^n,\ n \geqslant 2)$
$\overline{subset}$	the closure of *subset* in the topology of the larger set
∂D	the boundary of a set D
$\lvert D \rvert$	n-dimensional volume of the set D
$B_r(x)$	$\{y \in \mathbb{R}^n : \lvert y-x \rvert < r\}$
$\operatorname{supp}(f)$	the support of the function/distribution f
$d(x)$	distance function
c_p	$-((p-1)!\,\pi)^{-1}$
$\mathscr{L}^1(\Omega)$	Integrable functions over Ω
$\Gamma(z)$	$\Gamma_p(z) = c_p \bar{z}^{p-1} z^{-1}$ (Fundamental solution to $\partial^p/\partial \bar{z}^p$)
$\mathscr{A}_p(\Omega)$	$\{f \in \mathscr{L}^1(\Omega);\ \partial^p f/\partial \bar{z}^p = 0 \text{ in } \Omega\}$
$\mathscr{S}_{k,p}(\Omega, z^0)$	$\operatorname{span}\{G_k^{(i)}(\cdot, w);\ w \notin (\Omega \cup \{z^0\}),\ i = 0, \ldots, p-1\}$
$G_k(z, w)$	$z^k \Gamma(z-w)$
dA	The Lebesgue measure in the plane
$E_j(w)$	$\displaystyle\int_\Omega g(z)\, \Gamma^{(j)}(z-w)\, dA$
$\delta_y(x)$	Dirac measure concentrated at y
$\tilde{G}_k(z, w)$	$\displaystyle c_p(\bar{z}-\bar{w})^{p-1}\left(\frac{1}{(z-w)} - \sum_{j=0}^{k+p} \frac{w^j}{z^{j+1}}\right)$
$\tilde{\mathscr{S}}_{k,p}(\Omega)$	$\operatorname{span}\{\tilde{G}_k^{(i)}(\cdot, w);\ \forall w \notin \Omega : i = 0, 1, \ldots, p-1\}$
L_m	$\displaystyle L_m = \sum_{\lvert\alpha\rvert = m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$ uniformly elliptic
K	Fundamental solution to L_m
$K_s(x, y)$	$\displaystyle K_s(x, y, x^0) = K(x-y) - \sum_{\lvert\alpha\rvert < s} \frac{1}{\alpha!} K^{(\alpha)}(x^0 - y)(x - x^0)^\alpha$

$\tilde{K}_s(x, y)$	$\tilde{K}_s(x, y, x^0) = K(x-y) - \sum_{\lvert\alpha\rvert < s} \frac{1}{\alpha!} K^{(\alpha)}(x-x^0)(y-x^0)^\alpha$
$\mathfrak{F}$	$\mathfrak{F}(\Omega) = \{f \in \mathscr{L}^1(\Omega);\ L_m(f) = 0 \text{ in } \Omega\}$
$\mathfrak{F}^+$	$\mathfrak{F}^+(\Omega) = \{f \in \mathscr{L}^1(\Omega);\ L_m(f) \geqslant 0 \text{ in } \Omega\}$
$\mathfrak{F}_s(\Omega, x^0)$	$\operatorname{span}\{K_s^{(\alpha)}(\cdot, y);\ y \notin \Omega,\ \lvert\alpha\rvert \leqslant m-1\}$,
$\mathfrak{F}_s^+(\Omega, x^0)$	$\operatorname{span}^+\{\pm K_s^{(\alpha)}(\cdot, y);\ y \notin \Omega,\ \lvert\alpha\rvert \leqslant m-1 \text{ and } K_s(\cdot, y);\ y \in \Omega\}$
$\tilde{\mathfrak{F}}_s(\Omega, x^0)$	$\operatorname{span}^+\{\tilde{K}_s^{(\alpha)}(\cdot, y);\ y \notin \Omega,\ \lvert\alpha\rvert \leqslant m-1\}$
$\tilde{\mathfrak{F}}_s^+(\Omega, x^0)$	$\operatorname{span}^+\{\pm \tilde{K}_s^{(\alpha)}(\cdot, y);\ y \notin \Omega,\ \lvert\alpha\rvert \leqslant m-1 \text{ and } \tilde{K}_s(\cdot, y),\ y \in \Omega\}$
$U_s(y)$	$U_s^g(y) = \int_\Omega \tilde{K}_s(x, y)\, g(x)\, dx$
$\Lambda(z, R, D)$	$\bigcup_{x \in B(z, R) \cap D} \left\{ z + t \frac{(x-z)}{\lvert x-z\rvert};\ t \leqslant 1 \right\}$
Δ^p	p-times iterated Laplacian

0. INTRODUCTION

Consider a bounded domain $\Omega \subset \mathbb{C}$, fix a boundary point $z_0 \in \partial\Omega$, and let $\mathscr{L}^1(\Omega)$ denote the set of all functions integrable over Ω. Let also $\mathscr{A}_1(\Omega)$ denote the analytic functions in $\mathscr{L}^1(\Omega)$. Then it is well-known that $\mathscr{L}^1$-analytic functions that vanish continuously at z_0 (we denote this by $\mathscr{S}_1$) are dense in $\mathscr{A}_1(\Omega)$, provided z_0 is a non-isolated boundary point.

In the case of Ω being unbounded, one can reverse the above statement to the situation of rapidly decreasing functions at infinity. Namely, if the infinity point is a non-isolated boundary point of $\partial\Omega$, then the subset of $\mathscr{L}^1$-analytic functions that decrease as $O(\lvert x\rvert^{-k})$ at infinity, is dense in $\mathscr{A}_1(\Omega)$; here $k \geqslant 3$.

Now let $\Omega \in \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain. Let also L_m be a linear elliptic partial differential operator L_m of order m (an even positive integer). Then we want to analyze whether the subset of $\mathscr{L}^1$-solutions to L_m (in Ω), vanishing continuously at some boundary point $x^0 \in \partial\Omega$, is dense in the space of $\mathscr{L}^1$-solution to L_m (in Ω).

Similarly one can consider an unbounded domain Ω in $\mathbb{R}^N$, having the infinity as a boundary point. The question to raise, then, is whether the subclass of rapidly decreasing $\mathscr{L}^1$-solutions approximate all other solutions.

The possession of such an approximation property, for unbounded domains in $\mathbb{R}^N$, is heavily reliant on the thickness of the complement of the domain under consideration. It turns out that the problem has different features depending on the space dimension; e.g., for $N=2$ we show that such an approximation is possible if the infinity point is a non-isolated boundary point. However, for $N \geqslant 3$ the problem is quite different and by no means elementary. It appears to be connected with the problem of unique continuation for polynomial solutions to $L_m u = 0$.

The consideration of such a problem is motivated by two different problems. The first motivation is given by certain integral identities arising in some free boundary problems for the Laplacian or, if considered in $\mathbb{C}$, for the Cauchy operator (see [sh2, gs, sa2, ks2, ks3]). The second motivation comes from local and/or global estimates for solutions to certain (overdetermined) elliptic problems. Indeed, let Ω be an unbounded domain in $\mathbb{R}^N$ and suppose u satisfies (in the sense of distributions)

$$L_m u = g\chi_\Omega, \quad \text{in } \mathbb{R}^N \qquad u=0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \qquad |u(x)| \leqslant C(1+|x|^{m+k}),$$

where g is a given bounded function in $\mathbb{R}^N$, and k, C are positive constants. It is interesting, then, to know whether

$$u(x) \leqslant (1+|x|^m)(A \log(2+|x|)+B), \tag{0.1}$$

for some $A, B \geqslant 0$.

As we will show, this problem (when $A>0$) is strongly connected with the approximation problem studied here and its solve-ability, in turn, depends on the thickness of the complement of Ω near the infinity point. However, if we in (0.1) require $A=0$, then the problem becomes very hard and it is no longer naturally related to the approximation problem (see [ks1, hks]).

The approximation problem for both harmonic and analytic functions has been considered earlier by several people. For analytic functions see [be, sa1, sh1]; for harmonic and subharmonic functions see [sa2, ka, shg].

The plan of this paper is as follows. In Section 1, we will give the results for polyanalytic functions. Here we choose to give some proofs in detail, as they will be standard and repetitive technicalities in all future results. These kinds of detailed proofs will be omitted later.

In Section 2, we treat the problem in N-space dimension for uniformly elliptic operators. The local and global results, as well as the two dimensional and higher dimensional results, are separated. Finally, in Section 3, we discuss possible extensions and generalizations.

1. POLYANALYTIC FUNCTIONS IN $\mathbb{C}$

1.1. *Bounded Domains*

For any fixed integer $p \geqslant 1$ let $c_p = -1/(p-1)!\,\pi$, Ω be a bounded domain in $\mathbb{C}$, and

$$\Gamma(z) = \Gamma_p(z) = c_p \frac{\bar{z}^{p-1}}{z}.$$

Define

$$\mathscr{A}_p(\Omega) = \{ f \in \mathscr{L}^1(\Omega);\ \partial^p f/\partial \bar{z}^p = 0 \text{ in } \Omega \},$$

and for $z^0 \in \partial\Omega$, and $k \geqslant 0$ set

$$\mathscr{S}_{k,p}(\Omega, z^0) = \operatorname{span}\{ G_k^{(i)}(\cdot, w);\ \forall w \notin (\Omega \cup \{z^0\}),\ i = 0, ..., p-1 \},$$

where $G_k(z, w) = G_k(z, w, z^0) = (z - z^0)^k\, \Gamma(z - z^0 - w)$, and $f^{(i)} = \partial^i f/\partial \bar{z}^i$, for any i-times differentiable function f.

Observe that all functions in $\mathscr{S}_{k,p}(\Omega, z^0)$ vanish, with their derivatives of order $(k-1)$, at z^0.

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{C}$, and $z^0 \in \partial\Omega$ be a non-isolated boundary point. Then*

$$\overline{\mathscr{S}_{k,p}(\Omega, z^0)} = \mathscr{A}_p(\Omega), \qquad \forall k \geqslant 0,$$

where the closure is in $\mathscr{L}^1$-metric.

The proof of this theorem for the case $k = 0$ is due to L. Bers [be]. The general case is close to that of Bers. The main idea is the use of a family of cut-off functions previously used by Ahlfors–Bers.

We also refer to the works of A. O'Farrell [of1-2] and J. Verdera [v] for uniform approximation by polyanalytic functions and solutions of elliptic equations.

Proof of Theorem 1.1. We assume without loss of generality that $z^0 = 0$, and $k = 1$; the general case is proven similarly. It suffices, by the Hahn Banach theorem, to prove that any bounded linear functional (represented by an $\mathscr{L}^\infty$-function g) annihilating $\mathscr{S}_{1,p}$, also annihilates $\mathscr{A}_p(\Omega)$. We thus suppose that for $j = 0, ..., p-1$,

$$\int_\Omega g(z)\, G^{(j)}(z, w)\, dA = 0 \qquad \forall w \in \mathbb{C} \backslash (\Omega \cup \{0\}),$$

where dA is the Lebesgue measure in the plane. Next, using the continuity of the above integral in w (this is known and elementary to show) and the fact that the origin is a non-isolated boundary point, we let $w \to 0$ to obtain

$$\int_\Omega z\Gamma^{(j)}(z)\, g(z)\, dA = 0, \qquad (j=0, ..., p-1)$$

which amounts to

$$\int_\Omega \bar{z}^j g(z)\, dA = 0, \qquad \forall (j=0, ..., p-1). \tag{1.1}$$

Next for $w \in \mathbb{C} \setminus (\Omega \cup \{0\})$,

$$\begin{aligned} 0 &= \int_\Omega g(z)\, G^{(j)}(z, w)\, dA = \int_\Omega zg(z)\, \Gamma^{(j)}(z-w)\, dA \\ &= \int_\Omega (z-w)\, g(z)\, \Gamma^{(j)}(z-w)\, dA + w \int_\Omega g(z)\, \Gamma^{(j)}(z-w)\, dA \\ &= \sum_{i=0}^{p-1-j} C_i(-\bar{w})^{p-1-j-i} \int_\Omega \bar{z}^i g(z)\, dA + w \int_\Omega g(z)\, \Gamma^{(j)}(z-w)\, dA, \end{aligned}$$

where $C_i = \binom{p-1-j}{i}(-\pi(p-1-j)!)^{-1}$, and $j=0, ..., p-1$. Using (1.1) and dividing the above by w we will have that the functions

$$E_j(w) := \int_\Omega g(z)\, \Gamma^{(j)}(z-w)\, dA \qquad (j=0, ..., p-1)$$

defined in $\mathbb{C}$, vanish on $\mathbb{C} \setminus (\Omega \cup \{0\})$. Hence by continuity (since the origin is a non-isolated boundary point of $\partial\Omega$)

$$E_j(w) = 0 \qquad \forall j=0, ..., p-1, \qquad \text{and} \qquad w \in \mathbb{C} \setminus \Omega. \tag{1.2}$$

To complete the proof we need to show (and this is a standard procedure in approximation theory) that any annihilator of the fundamental solution $\Gamma(\cdot - w)$ (where $w \notin \Omega$) along with its derivatives up to order $(p-1)$ is also an annihilator of all $\mathscr{A}_p(\Omega)$. For this purpose let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence of C^∞-functions with support in Ω and with the property that $\omega_n = 0$ in a neighborhood of $\partial\Omega$, $\lim_{n\to\infty} \omega_n(z) = 1$ for $z \in \Omega$, and for $|\alpha| \geqslant 1$,

$$|D^\alpha \omega_n(z)| \leqslant \frac{C_\alpha}{n} (d(z))^{-|\alpha|}\, |\log d(z)|^{-1}, \tag{1.3}$$

where $d(z) = \operatorname{dist}(z, \partial\Omega)$, α is multi-index and D^α is the partial derivative with respect to the real variables x_1, x_2 $(z = x_1 + ix_2)$. For existence of such functions we refer to [ah]; see also [be] and [kr]. By (1.2) and the definition of the function E_0, we will have

$$\frac{\partial^p E_0}{\partial \bar{z}^p}(z) = g \qquad \text{in } \Omega, \tag{1.4}$$

in the sense of distributions, and also

$$\lim_{\Omega \ni z \to \partial\Omega} \frac{\partial^j E_0}{\partial \bar{z}^j}(z) = 0, \qquad (j = 0, ..., p-1).$$

Now using this and observing that $E_j(z) = (-1)^j E_0^{(j)}(z)$, we may deduce that for $z \in \Omega$

$$|E_j(z)| \leqslant (d(z))^{p-j-1} \, \|E_{p-1}(z)\|_\infty, \qquad (j = 0, ..., p-2) \tag{1.5}$$

where $\|\cdot\|_\infty$ means the supremum norm, and it is taken on the segment $[z, z']$; with z' being the nearest point, on $\partial\Omega$, to z. Now well-known estimates for the Cauchy's integral (see [kr]) implies

$$|E_{p-1}(z)| \leqslant C d(z) \, |\log d(z)|, \qquad (C > 0, \text{ fixed})$$

which in conjunction with (1.5) implies

$$|E_j(z)| \leqslant C (d(z))^{p-j} \, |\log d(z)|, \qquad (j = 0, ..., p-1). \tag{1.6}$$

Next let $f \in \mathscr{A}_p(\Omega)$. Then, using (1.4) and the cutoff functions ω_n, we have

$$\int_\Omega fg \, dA = \lim_n \int_\Omega fg\omega_n \, dA = \lim_n \int_\Omega f \frac{\partial^p E_0}{\partial \bar{z}^p} \omega_n \, dA.$$

By the standard Leibnitz formula we obtain

$$\int_\Omega f \frac{\partial^p E_0}{\partial \bar{z}^p} \omega_n \, dA = \int_\Omega f \frac{\partial^p}{\partial \bar{z}^p} (E_0 \omega_n) \, dA - \sum_{j=0}^{p-1} \binom{p}{j} (-1)^j \int_\Omega f E_j \omega_n^{(p-j)} \, dA. \tag{1.7}$$

where we have used $E_j(z) = (-1)^j E_0^{(j)}(z)$. Now the first integral to the right side in (1.7) vanishes, through integration by parts, since $f \in \mathscr{A}_p$, and the rest of the terms are bounded (according to (1.3) and (1.6)) by

$$\frac{C}{n} \sum_{j=0}^{p-1} \int_\Omega |f(z)| \, (d(z))^{p-j} \, |\log d(z)| \frac{dA}{(d(z))^{p-j} \, |\log d(z)|}, \qquad (C > 0, \text{ fixed})$$

which tends to zero as $n \to \infty$. Hence

$$\int_\Omega fg\, dA = 0 \qquad \forall f \in \mathcal{A}_p.$$

This completes the proof. ∎

EXAMPLE 1.1. This example concerns the sharpness of the above result. Let $\alpha = (\alpha_0, ..., \alpha_{p-1})$ be the solution to $\mathcal{M}\alpha = b$, where $\mathcal{M}$ is the $p \times p$ matrix with entries $a_{mn} = \binom{p+n}{m}$, $(m, n = 0, ..., p-1)$ and the vector $b = (b_0, ..., b_{p-1})$ has the components $b_m = \binom{p-1}{m}$.

Let us define

$$v(z) := c_p \left(\sum_{j=0}^{p-1} \alpha_j \, |z|^{2j} \, \bar{z}^p - \frac{\bar{z}^{p-1}}{z} \right)$$

$$h(z) := c_p \left(\sum_{j=0}^{p-1} \alpha_j (p+j) \cdots (1+j) \, |z|^{2j} \right).$$

Then one may easily verify that

$$\frac{\partial^p u}{\partial \bar{z}^p}(z) = h(z) - \delta_0 \qquad \text{in } B(0, 1), \qquad \text{(as distributions)}$$

$$\frac{\partial^j u}{\partial \bar{z}^j}(z) = 0 \qquad \text{on } \partial B(0, 1), \qquad (j = 0, ..., p-1)$$

where δ_0 is the Dirac mass concentrated at the origin.

To this end, for fixed $r > 0$ and $z^0 \in \mathbb{C}$, we define $u_r(z) := r^p u((z - z^0)/r)$, and $h_r(z) = h((z - z^0)/r)$. Then one readily verifies

$$\begin{aligned} &\frac{\partial^p u_r}{\partial \bar{z}^p}(z) = h_r(z) - \delta_0 \qquad \text{in } B(z^0, r), \qquad \text{(as distributions)} \\ &\frac{\partial^j u_r}{\partial \bar{z}^j}(z) = 0 \qquad \text{on } \partial B(z^0, r) \qquad (j = 0, ..., p-1). \end{aligned} \tag{1.8}$$

Suppose now Ω is a bounded domain and $z^0 \in \partial\Omega$ is an isolated boundary point. Hence there exists $r > 0$ such that $B(z^0, r) \subset \Omega \cup \{z^0\}$. We define $g_r(z) = h_r(z)\, \chi_{B(z^0, r)}$ and the bounded linear functional

$$T\colon f \to \int_\Omega f g_r \, dA,$$

on $\mathscr{L}^1(\Omega)$. Then for $f \in \mathscr{A}_p(\Omega)$, and continuous up to z^0, we can use integration by parts to obtain, in combination with (1.8),

$$\int_{\Omega} f g_r \, dA = \int_{B(z^0, r)} f h_r \, dA = \int_{B(z^0, r)} f \left(\frac{\partial^p u_r}{\partial \bar{z}^p} + \delta_0 \right) dA = f(z^0).$$

It thus follows that the bounded linear functional T, vanishes on $\mathscr{S}_{1,p}(\Omega, z^0)$ but not on $\mathscr{A}_p(\Omega)$.

1.2. *Unbounded Domains*

Throughout this section we will assume that $B(0, 1) \cap \Omega = \emptyset$. This geometrical restriction is, indeed, a technicality and can be removed by means of some new techniques of [km]; we omit discussing this matter. Let now $\infty \in \partial\Omega$ be a non-isolated boundary point, i.e. there exists a sequence $\{z_j\} \subset \partial\Omega$, with $\lim_j |z_j| = \infty$. We will prove that all $\mathscr{L}^1$-polyanalytic functions in Ω can be approximated by the subset of rapidly decreasing ones at the infinity. Indeed, let

$$\tilde{G}_k(z, w) = c_p(\bar{z} - \bar{w})^{p-1} \left(\frac{1}{(z-w)} - \sum_{j=0}^{k+p} \frac{w^j}{z^{j+1}} \right).$$

Then, using the fact that the sum above is the $(k+p)$-th Taylor polynomial of $(z-w)^{-1}$ near $w = 0$, as a function of w, we obtain for each fixed w

$$\tilde{G}_k^{(i)}(z, w) \leqslant \frac{C\, |w|^{(k+p+1-i)}}{|z|^{k+3}} \qquad \forall\, |z| > 2\, |w|.$$

Since also $B(0, 1) \cap \Omega = \emptyset$, we conclude $\tilde{G}_k(z, w) \in \mathscr{L}^1(\Omega)$ as a function of z, for all $k \geqslant 0$, and $w \in \mathbb{C}$. It is also easy to see that $\tilde{G}_k$ is a fundamental solution to $\partial^p/\partial\bar{z}^p$ for $w, z \neq 0$. More precisely $\partial^p \tilde{G}_k(z, w)/\partial\bar{z}^p = \delta_w(z)$, for $w, z \neq 0$.

Remark 1.1. Let $g \in \mathscr{L}^\infty(\mathbb{R}^N)$, and suppose $\Omega \cap B(0, 1) = \emptyset$. Define

$$v(w) = \int_{\Omega} g(z)\, \tilde{G}_k(z, w)\, dA.$$

Then it follows that $\partial^p v/\partial\bar{w}^p = g\chi_\Omega$, and hence $\tilde{v} := \partial^{p-1} v/\partial\bar{w}^{p-1}$ satisfies $\partial\tilde{v}/\partial\bar{w} = g\chi_\Omega$. Indeed, v is the Cauchy integral of $g\chi_\Omega$. Now simple calculations (see [kr; Section 3, Lemma 1.4]) reveals $|\tilde{v}(w)| \leqslant C(|w|+1)\log(|w|+2)$, and hence $|\partial^j v/\partial\bar{w}^j| \leqslant C(|w|+1)^{p-j}\log(|w|+2)$.

Next define

$$\widetilde{\mathscr{S}}_{k,\,p}(\Omega) := \mathrm{span}\{\widetilde{G}_k^{(i)}(\cdot, w);\ \forall w \notin \Omega : i = 0, 1, ..., p-1\}.$$

Then we claim the following.

THEOREM 1.2. *For any unbounded domain $\Omega \subset \mathbb{C}$, with $B(0, 1) \cap \Omega = \emptyset$, there holds*

$$\overline{\widetilde{\mathscr{S}}_{k,\,p}(\Omega)} = \mathscr{A}_p(\Omega), \qquad \forall k \geqslant 0, \tag{1.9}$$

provided the infinity point is a non-isolated boundary point.

Proof. First suppose $k \geqslant 1$. Then, as in the proof of Theorem 1.1, let $g \in \mathscr{L}^\infty(\Omega)$ represent the linear functional which is zero on $\widetilde{\mathscr{S}}_{k,\,p}$, i.e. for $(i = 0, 1, ..., p-1)$

$$\int_\Omega g(z)\, \widetilde{G}_k^{(i)}(z, w)\, dA = 0, \qquad \forall w \notin \Omega. \tag{1.10}$$

In order to prove that g also annihilates the whole space $\mathscr{A}_p$ we show first that for $i = 0, 1, ..., p-1$,

$$\widetilde{E}_{i,\,k-1}(w) := \int_\Omega g(z)\, \widetilde{G}_{k-1}^{(i)}(z, w)\, dA,$$

vanishes on $\mathbb{C} \backslash \Omega$.

Now rewriting (1.10) we have

$$\int_\Omega g(z)\, \widetilde{G}_{k-1}^{(i)}(z, w)\, dA$$
$$= \sum_{s=0}^{p-1-i} C_s(-\bar{w})^{p-1-i-s}\, w^{k+p} \int_\Omega \frac{g(z)\, \bar{z}^s}{z^{k+p+1}}\, dA, \qquad \forall w \notin \Omega,$$

where $C_s = \binom{p-1-i}{s}(-\pi(p-1-i)!)^{-1}$. Dividing the above by $w^{2p-1-i-l+k}$ (with $l = 0$) and letting $w \to \infty$ (through $\mathbb{C} \backslash \Omega$) we will have

$$0 = \int_\Omega \frac{g(z)\, \bar{z}^l}{z^{k+p+1}}\, dA, \qquad (\text{for } l = 0).$$

Repeating this argument for $l = 2, 3, ...p-1$, gives the desired result. Hence $\widetilde{E}_{i,\,k-1}(z)$ vanishes on $\mathbb{C} \backslash \Omega$ for $i = 0, 1, ..., p-1$. In the same vein, we can show that $\widetilde{E}_{i,\,k-2}, ..., \widetilde{E}_{i,\,0}$ vanish on $\mathbb{C} \backslash \Omega$ for $i = 0, 1, ..., p-1$.

Now $\widetilde{E}_{0,\,0}$ plays the same role as $E_0(z)$ did in Theorem 1.1. Thus, to complete the proof one has to repeat the argument in the proof of Theorem 1.1, using cutoff functions ω_n. However, as the domain is unbounded, we need also to cut off the domain so that it becomes bounded. To this end, one introduces a different type of cutoff functions ρ_m (having compact support), which in conjunction with Remark 1.1 will bring the proof into a completion. As it might become repetitive we leave the details of this part to the reader (see [sa1, Lemma 3], for $p=1$). ∎

EXAMPLE 1.2. Let Ω be any domain in $\mathbb{C}$ with a nice (analytic) boundary. A function $S(z)$ is said to be the Schwarz function of $\partial\Omega$ if it is analytic in an interior neighborhood of $\partial\Omega$ and $S(z)=\bar{z}$ on $\partial\Omega$; see [sh2]. In order to generalize this notion for p-analytic functions we define S_p to be the p-Schwarz function for $\partial\Omega$ if it is p-analytic in an interior neighborhood of $\partial\Omega$ and if $(\partial^j/\partial\bar{z}^j)(S(z)-\bar{z}^p)=0$, for $j=0, ..., p-1$, and $z\in\partial\Omega$. Let now S be the Schwarz function of $\partial\Omega$ then one may easily verify, for appropriate choices of $\{\beta_j\}$,

$$S_p(z)=\sum_{j=1}^{p} \beta_j \bar{z}^{p-j} S^j(z),$$

is the p-Schwarz function of $\partial\Omega$. It is also noteworthy that

$$\operatorname{supp}\left(\frac{\partial^p}{\partial\bar{z}^p} S_p(z)\right)\subset \operatorname{supp}\left(\frac{\partial}{\partial\bar{z}} S(z)\right),$$

where supp denotes the support of the function/distribution, and

$$p\cdots(p-j)\, S_{p-j}=\frac{\partial^j S_p}{\partial\bar{z}^j}.$$

The latter in turn implies that if S_1 (the Schwarz function) grows as $|z|^m$ then S_p grows like $|z|^{p+m-1}$. We will use this fact in the next example where we show that in Theorem 1.2, the assumption that the infinity point is a non-isolated boundary point is indispensable, and hence Theorem 1.2 is sharp.

EXAMPLE 1.3. Let $m\geqslant 1$ be an integer. Then, according to an example of B. Gustafsson (see [sh3, Theorem 2]) there is a domain D in $\mathbb{C}$ with analytic boundary, and bounded (and connected) complement. Moreover, D admits a Schwarz function $S(z)$, which is analytic in D, and which grows like $|z|^m$ at ∞. Hence by Example 1.2 we will have that the p-Schwarz function of ∂D exists and is p-analytic in D with growth $p+m-1$ at ∞. Now by defining $u(z)$ as

$$u(z)=(\bar{z}^p-S_p(z))/p!$$

we may conclude, by the above discussion, that for any non-negative integer k there is a function $u(z)$ and a domain D such that for $j=0, ..., p-1$

$$\begin{aligned}
&\frac{\partial^p u}{\partial \bar{z}^p}(z)=1 && \text{in } D \\
&\frac{\partial^j u}{\partial \bar{z}^j}(z)=0 && \text{on } \partial D, \\
&\sup_{|z| \leqslant R} |u(z)| \approx C_R R^{p+k} && \text{large } R,
\end{aligned} \tag{1.11}$$

where $0 < C_0 \leqslant C_R \leqslant C_1 < \infty$ for some constants C_0, C_1.

Let now Ω be any domain with bounded complement. Suppose, $z^0 \in \mathbb{C} \backslash \Omega$ and define, by translation and scaling,

$$u_r(z)=u(r(z-z^0))/r^p, \qquad D_r := \{z \colon r(z-z^0) \in D\},$$

were $r>0$ is taken small enough to guarantee $D_r \subset \Omega$. In order to show the sharpness of Theorem 1.2, let the domain Ω have the property that $\overline{\tilde{\mathscr{S}}_{k,p}(\Omega)}=\mathscr{A}_p(\Omega)$, for some $k>0$. Also let u_r be as above with growth $k+p$, and $g=\chi_{D_r} \in \mathscr{L}^\infty(\Omega)$. Then by Greens theorem and (1.11) one can show

$$\int_\Omega \tilde{G}_k(z, w)\, g(z)\, dA=0, \qquad \forall w \notin \Omega.$$

Now by the assumption that $\overline{\tilde{\mathscr{S}}_{k,p}(\Omega)}=\mathscr{A}_p$ we conclude that

$$\int_\Omega \tilde{G}_0(z, w)\, g(z)\, dA=\int_{D_r} \tilde{G}_0(z, w)\, dA=0, \qquad \forall w \notin \Omega,$$

and by unique continuation ($\mathbb{C} \backslash D_r$ is connected) also on $\mathbb{C} \backslash D_r$.

Now define

$$u_r'(w)=\int_{D_r} \tilde{G}_0(z, w)\, g(z)\, dA \qquad \forall w \in \mathbb{C}.$$

Then $\partial^p u_r'/\partial \bar{z}^p=\chi_{D_r}$ in $\mathbb{C}$, and $u_r'=0$ in $\mathbb{C} \backslash D_r$. Hence

$$\frac{\partial^p (u_r-u_r')}{\partial \bar{z}^p}=\chi_{D_r}-\chi_{D_r}=0 \qquad \text{in } \mathbb{C},$$

and $u_r - u'_r = 0$ in $\mathbb{C}\backslash D_r$. Hence by the unique continuation $u'_r = u_r$. Next using standard estimates for the Cauchy's integral (see [be]) we will have

$$|u_r(z)| = |u'_r(z)| \leqslant \int_{D_r} |G_0(w, z)\, g(w)|\, dA \leqslant C(|z|+1)^p \log(|z|+2). \qquad (1.12)$$

Hence if we choose $k \geqslant 1$, then by (1.11), u_r has approximately a growth of order $p+1$, contradicting (1.12). This shows that $\overline{\widetilde{\mathscr{S}}_{k,\,p}(\Omega)} \neq \mathscr{A}_p(\Omega)$.

2. HOMOGENEOUS ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

2.1. *Generalized Kernels*

In $\mathbb{R}^N$, consider a homogeneous linear elliptic partial differential operator L_m of (even) order m, and with constant coefficients

$$L_m = \sum_{|\alpha| = m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha},$$

where

$$\sum_{|\alpha| = m} a_\alpha \xi^\alpha \neq 0 \qquad \text{for} \quad \xi \in \mathbb{R}^N \quad \text{and} \quad |\xi| \neq 0.$$

Let now K be the usual fundamental solution for L_m with singularity at the origin. Then K can be expressed as follows.

(i) If N is odd or if N is even and $N > m$, then

$$K(x) = |x|^{m-N} H_0(x),$$

where $H_0(tx) = H_0(x)$ if $x \neq 0$, $t > 0$.

(ii) If N is even and $N \leqslant m$, then

$$K(x) = |x|^{m-N} H_0(x) + H_1(x) \log |x|,$$

where H_1 is a homogeneous polynomial of degree $(m - N)$, (see [jo]).

We also define two different fundamental solutions "generalized kernels" K_s and $\widetilde{K}_s$ as

$$K_s(x, y) = K_s(x, y, x^0) = K(x-y) - \sum_{|\alpha| < s} \frac{1}{\alpha!} K^{(\alpha)}(x^0 - y)(x - x^0)^\alpha, \qquad s > 0,$$

$$\widetilde{K}_s(x, y) = \widetilde{K}_s(x, y, x^0) = K(x-y) - \sum_{|\alpha| < s} \frac{1}{\alpha!} K^{(\alpha)}(x - x^0)(y - x^0)^\alpha, \qquad s > 0,$$

$$K_0(x, y) = K_0(x, y, x^0) = K(x-y),$$

where $K^{(\alpha)}(x)=\partial^{(\alpha)}K(x)/\partial x^{\alpha}$. It is easy to see that K_s and $\tilde{K}_s$ satisfy the following:

$$
\begin{aligned}
&L_m K_s(x, y)=\delta_y(x), && (y \neq x^0)\\
&L_m \tilde{K}_s(x, y)=\delta_y(x), && (x \neq x^0)\\
&K_s^{(\alpha)}(x^0, y)=0, && \forall\, |\alpha|<s,\\
&|\tilde{K}_s^{(\alpha)}(x, y)| \leqslant C\,\frac{|y-x^0|^s(A_1+B_1\log|y-x^0|)}{|x-x^0|^{|\alpha|+s+N-m}}, && \forall\alpha, \quad |x|\geqslant 2\,|y|,
\end{aligned}
$$

where δ is the Dirac measure, $A_1=1$, $B_1=0$ for N odd or for $N>m$, and $A_1=0$ and $B_1=1$ for N even and $N\leqslant m$.

Obviously for a given domain Ω, $\tilde{K}_s$ is not necessarily in $\mathscr{L}^1(\Omega)$, whatever s may be. This situation happens if the domain is unbounded and $x^0\in\bar{\Omega}$. Indeed, $\tilde{K}_s$ is locally non-integrable, when $s>m$, and globally non-integrable when $s\leqslant m$. To overcome this difficulty we assume that there is a constant $r>0$ such that $B(x^0, r)\subset\mathbb{R}^N\backslash\Omega$. This assumption will assure the local integrability for any s and by choosing $s>m$ we will have the global integrability. In the sequel it is tacitly understood that if we consider the kernels $\tilde{K}_s$ on an unbounded domain Ω, with $\bar{\Omega}\neq\mathbb{R}^N$, then x^0 is chosen in the interior of $\mathbb{R}^N\backslash\bar{\Omega}$. So that there will be no problem with the local integrability of $\tilde{K}_s$. We are thus forced to assume $\bar{\Omega}\neq\mathbb{R}^N$. This will also be assumed through the rest of the paper. For a thorough treatment of the Laplacian case we refer to the paper of L. Karp [ka].

For a given domain Ω, we set

$$
\begin{aligned}
\mathfrak{F} &:= \mathfrak{F}(\Omega)=\{f\in\mathscr{L}^1(\Omega);\ L_m(f)=0 \text{ in } \Omega\},\\
\mathfrak{F}^+ &:= \mathfrak{F}^+(\Omega)=\{f\in\mathscr{L}^1(\Omega);\ L_m(f)\geqslant 0 \text{ in } \Omega\}.
\end{aligned}
$$

If Ω is bounded with $x^0\notin\Omega$ we set

$$
\begin{aligned}
\mathfrak{F}_s &:= \mathfrak{F}_s(\Omega, x^0)=\operatorname{span}\{K_s^{(\alpha)}(\cdot, y),\ y\notin\Omega,\ |\alpha|\leqslant m-1\},\\
\mathfrak{F}_s^+ &:= \mathfrak{F}_s^+(\Omega, x^0)=\operatorname{span}^+\{\pm K_s^{(\alpha)}(\cdot, y),\\
&\qquad y\notin\Omega,\ |\alpha|\leqslant m-1 \text{ and } K_s(\cdot, y),\ y\in\Omega\},
\end{aligned}
$$

where $K_s^{(\alpha)}(\cdot, y)=K_s^{(\alpha)}(\cdot, y, x^0)$. For unbounded Ω with $x^0\in\mathbb{R}^N\backslash\bar{\Omega}$, we define

$$
\begin{aligned}
\tilde{\mathfrak{F}}_s &:= \tilde{\mathfrak{F}}_s(\Omega, x^0)=\operatorname{span}\{\tilde{K}_s^{(\alpha)}(\cdot, y),\ y\notin\Omega,\ |\alpha|\leqslant m-1\},\\
\tilde{\mathfrak{F}}_s^+ &:= \tilde{\mathfrak{F}}_s^+(\Omega, x^0)=\operatorname{span}^+\{\pm \tilde{K}_s^{(\alpha)}(\cdot, y),\\
&\qquad y\notin\Omega,\ |\alpha|\leqslant m-1 \text{ and } \tilde{K}_s(\cdot, y),\ y\in\Omega\},
\end{aligned}
$$

where $K_s^{(\alpha)}(\cdot, y) = K_s^{(\alpha)}(\cdot, y, x^0)$, and span (span$^+$) means all finite linear combinations (with positive coefficients). In addition to these we define the "generalized potential" U_s for an unbounded domain Ω with $x^0 \notin \bar{\Omega}$, and an $\mathscr{L}^\infty$ function g in Ω, by

$$U_s(y) = U_s^g(y, x^0) := \int_\Omega \tilde{K}_s(x, y, x^0)\, g(x)\, dx.$$

The following results will be needed for our main theorems.

THEOREM 2.1. *The generalized potential $U_s = U_s^g$, with bounded g, satisfies*

$$U_{m+1}^{(\alpha)}(y) \leqslant C(|y|+1)^{m-|\alpha|} \log(|y|+2), \qquad \forall\, |\alpha| < m.$$

A proof of this theorem for the case of Laplacian can be found in [ka]. For L_m as above one may use similar techniques as that of [ka] to prove Theorem 2.1.

THEOREM 2.2. *Let Ω_1 be a bounded domain and Ω_2 an unbounded one in $\mathbb{R}^N$. Suppose also $\bar{\Omega}_2 \neq \mathbb{R}^N$, and let $x^0 \notin \bar{\Omega}_2$. Then*

$$\overline{\mathfrak{F}_0(\Omega_1)} = \mathfrak{F}(\Omega_1), \qquad \overline{\mathfrak{F}_0^+(\Omega_1)} = \mathfrak{F}^+(\Omega_1),$$
$$\overline{\tilde{\mathfrak{F}}_{m+1}(\Omega_2, x^0)} = \tilde{\mathfrak{F}}(\Omega_2) \qquad \overline{\tilde{\mathfrak{F}}_{m+1}^+(\Omega_2, x^0)} = \tilde{\mathfrak{F}}^+(\Omega_2).$$

We will not give a proof of this theorem. For the Laplace operator we refer to [sa] (unbounded domains in $\mathbb{R}^2$), [sa] (bounded domains in $\mathbb{R}^N$) and [ka] (unbounded domains in $\mathbb{R}^N$). For general operators similar techniques as those in ([sa, ka]) work.

2.2. *Bounded Domains*

THEOREM 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, and x^0 a non-isolated point of $\mathbb{R}^N \backslash \Omega$. Then for $s \geqslant 0$*

$$\overline{\mathfrak{F}_s^+(\Omega, x^0)} = \mathfrak{F}^+(\Omega), \qquad \overline{\mathfrak{F}_s(\Omega, x^0)} = \mathfrak{F}(\Omega).$$

Proof. We only give the proof for the class $\mathfrak{F}_1$. For $s > 1$ or the class $\mathfrak{F}_s^+$, a similar argument works. Also for $s = 0$ Theorem 2.2 applies. Let us again consider a functional, which is zero on $\mathfrak{F}_1$, and denote by g the $\mathscr{L}^\infty$-function that represents the functional. Then, for $|\alpha| < m$ and $y \notin \Omega \cup \{x^0\}$,

$$0 = \int_\Omega K_1^{(\alpha)}(x, y)\, g(x)\, dx = \int_\Omega K_0^{(\alpha)}(x, y)\, g(x)\, dx - K_0^{(\alpha)}(x^0, y) \int_\Omega g(x)\, dx. \tag{2.1}$$

Since $|K_0^{(\alpha)}(x^0, y)| \approx C\,|x^0 - y|^{-|\alpha| - N + m}\,(A_1 + B_1 \log|x^0 - y|)$, (where $A_1 = 1$, $B_1 = 0$ for N odd or for $N > m$, and $A_1 = 0$ and $B_1 = 1$ for N even and $N \leqslant m$) we take $|\alpha| = m - 1$, multiply both sides of (2.1) by $|x^0 - y|^{N-1}\,(A_1 + B_1 \log|x^0 - y|)$, and let $y \to x^0$, through $\mathbb{R}^N \backslash \Omega$ (this is possible since x^0 is a non-isolated point of $\mathbb{R}^N \backslash \Omega$), to obtain

$$\int_\Omega g(x)\,dx = 0.$$

Putting this into (2.1) we will have

$$\int_\Omega K_0^{(\alpha)}(x, y)\,g(x)\,dx = 0 \qquad \forall\ y \notin \Omega \cup \{x^0\}, \quad |\alpha| = m - 1,$$

and by continuity also at x^0. Repeating this argument for $|\alpha| = m - 2$, $m - 3, ..., 0$, we wind up with

$$\int_\Omega K_0^{(\alpha)}(x, y)\,g(x)\,dx = 0 \qquad \forall y \notin \Omega \quad \text{and} \quad |\alpha| \leqslant m - 1.$$

Hence g annihilates $\mathfrak{F}_0$. The result now follows by applying Theorem 2.2. ▮

Example 2.1. Theorem 2.3 is sharp, in the sense that the boundary point in question has to be non-isolated. Indeed, let x^0 be an isolated boundary point. Then there is $r > 0$ such that $B(x^0, r) \subset \Omega \cup \{x^0\}$. Now take $g(x) = \chi_{B(x^0, r)}$ then for any subharmonic function f on Ω (and continuous on $\bar{\Omega}$)

$$\int_\Omega f(x)\,g(x)\,dx = \int_{B(x^0, r)} f(x)\,dx \geqslant |B(x^0, r)|\,f(x^0), \tag{2.2}$$

where $|B(x^0, r)|$ means the volume of the ball $B(x^0, r)$. Now let L_m be the Laplacian, and $\mathfrak{F}^+$ be the class of integrable subharmonic functions on Ω. Then (2.2) gives that the functional

$$T: f \to \int_\Omega f(x)\,g(x)\,dx,$$

is nonnegative on $\mathfrak{F}_1^+(\Omega, x^0)$. Whence for $f \equiv -1$ $(\in \mathfrak{F}^+(\Omega))$,

$$T(f) = \int f(x)\,g(x)\,dx = \int_{B(x^0, r)} f\,dx = -|B(x^0, r)| < 0.$$

This proves the sharpness of the theorem.

EXAMPLE 2.2. It is not easy to give examples similar to that of Example 2.1 for general operators. The problem is to find a function u satisfying

$$(L_m(u)+\delta_0)=:g\in\mathscr{L}^\infty(B(0,1)), \qquad u^{(\alpha)}=0 \quad \text{on } \partial B(0,1), \qquad |\alpha|\leqslant m-1.$$

Having proven the existence of such a function u, one repeats the argument in Example 1.1. However, for p-times iterated Laplacian $L_m=\Delta^p$ $(m=2p)$, one can give an explicit expression for u. Namely,

$$u(x)=\left(\sum_{j=0}^{2p-1}\alpha_j\,|x|^{2j}-K(x)\right),$$

where K is the fundamental solution for Δ^p, and α_j are chosen appropriately to ensure the boundary condition $u^{(\alpha)}=0$ on $\partial B(0,1)$ $(|\alpha|\leqslant m-1)$.

2.3. *Unbounded Domains in* $\mathbb{R}^2$

Let us now consider an unbounded domain Ω in the plane. We will pay attention to situations where rapidly decreasing $\mathscr{L}^1$-solutions (at infinity) of L_m will approximate all other solutions.

THEOREM 2.4. *Let Ω be an unbounded domain in $\mathbb{R}^2$, with $x^0\notin\bar{\Omega}$. Suppose, moreover, the infinity point is a non-isolated boundary point. Then, for $s\geqslant m+1$,*

$$\overline{\tilde{\mathfrak{F}}_s(\Omega,x^0)}=\tilde{\mathfrak{F}}(\Omega).$$

Proof. As before, for an annihilator $g\in\mathscr{L}^\infty(\Omega)$ of $\tilde{\mathfrak{F}}_s$ let $U_s=U_s^g$ be the generalized potential with density g. Then

$$L_m(U_s)=g\chi_\Omega \quad \text{in } \mathbb{R}^2, \qquad \text{and} \qquad U_s^{(\alpha)}=0 \quad \text{on } \mathbb{R}^2\backslash\Omega, \tag{2.3}$$

for $|\alpha|\leqslant m-1$. Now one may rewrite U_s as

$$U_s(y)=U_{m+1}(y)-\sum_{i=m+1}^{s-1}P_i(y), \tag{2.4}$$

where

$$P_i(y)=\sum_{|\alpha|=i}b_\alpha(y-x^0)^\alpha,$$

with

$$b_\alpha=\frac{1}{\alpha!}\int_\Omega K^{(\alpha)}(x-x^0)\,g(x)\,dx.$$

and $L_m(P_i)=0$ in $\mathbb{R}^2$. Now in view of Theorem 2.2 and expression (2.4) it suffices to show $P_i \equiv 0$ for all $m+1 \leqslant i \leqslant s-1$.

By the assumption that the infinity point is a non-isolated boundary point there is a sequence $\{y^j\}$ such that $y^j \in \partial\Omega$ and $|y^j| \to \infty$. Thus, by the boundary conditions in (2.3), for all such y^j we have

$$U_{m+1}^{(\alpha)}(y^j) = \sum_{i=m+1}^{s-1} P_i^{(\alpha)}(y^j), \qquad \forall\, |\alpha| \leqslant m-1. \tag{2.5}$$

Now let $z^j = y^j/|y^j|$ be the projection of y^j on the unit sphere. Then for a subsequence (which we relabel as the original sequence) z^j converges to a point z^0 on the unit sphere. Dividing both sides of (2.5) by $|y^j|^{s-|\alpha|-1}$ $(s-1 \geqslant m+1)$ and letting j tend to the infinity we obtain, using Theorem 2.1 and the homogeneity of $P_i^{(\alpha)}$,

$$P_{s-1}^{(\alpha)}(tz^0)=0, \qquad \forall t \in \mathbb{R}, \qquad \text{and} \qquad |\alpha| \leqslant m-1.$$

Hence P_{s-1} satisfies

$$L_m(P_{s-1})=0, \qquad \text{and} \qquad P_{s-1}^{(\alpha)}=0 \quad \text{on } l_0, \qquad \forall\, |\alpha| \leqslant m-1,$$

where $l_0=\{tz^0,\, t \in \mathbb{R}\}$. Hence, by the Cauchy Kowalewski theorem, $P_{s-1} \equiv 0$. Similarly one can prove $P_i \equiv 0$ for all $m+1 \leqslant i \leqslant s-2$. This proves the theorem. ∎

2.4. *Unbounded Domains in* $\mathbb{R}^N$.

The situation for unbounded domains in $\mathbb{R}^N$, for $N \geqslant 3$, is rather different from that of $\mathbb{R}^2$ and also more complicated. In this case, the Cauchy–Kowalewski theorem (see the final step in Theorem 2.4) does not apply anymore, unless the limit set of the scaling of $\mathbb{R}^N \backslash \Omega$ is an $(N-1)$-dimensional analytic hypersurface. This forces us to use another kind of uniqueness theorem. For second degree operators there are uniqueness theorems, expressed in terms of capacity-density (see Condition B below), which gives us the desired approximation theorem with very weak hypotheses. The lack of such results for higher degree operators, undermines similar conclusion for L_m with $m>2$. However, we can still state some results with stronger assumptions.

For any subset D of $\mathbb{R}^N$ we define $\Lambda(z, R, D)$ to be the truncated cone

$$\Lambda(z, R, D) := \bigcup_{x \in B(z, R) \cap D} \left\{ z + t\,\frac{(x-z)}{|x-z|};\; t \leqslant 1 \right\}.$$

Condition A. A given domain Ω is said to satisfy Condition A if there is a sequence of points $\{y^j\}$ in $\mathbb{R}^N \backslash \Omega$, positive numbers $R_j \to \infty$, and $\varepsilon > 0$, such that $B(y^j, \varepsilon) \cap \Omega = \varnothing$ and

$$\frac{|A(y^j, R_j, \mathbb{R}^N \backslash \Omega)|}{|B(0, R_j)|} \geqslant \varepsilon, \qquad \forall j.$$

Condition B. Similarly a given domain Ω is said to satisfy Condition B if there is a sequence of points $\{y^j\}$ in $\mathbb{R}^N \backslash \Omega$, positive numbers $R_j \to \infty$, and $\varepsilon > 0$ such that $B(y^j, \varepsilon) \cap \Omega = \varnothing$ and

$$\frac{\operatorname{cap}(A(y^j, R_j, \mathbb{R}^N \backslash \Omega))}{\operatorname{cap}(B(0, R_j))} \geqslant \varepsilon, \qquad \forall j,$$

where $\operatorname{cap}(D)$ denotes the capacity of the set D in $\mathbb{R}^N$, see [la].

LEMMA 2.5. *Let P_k be a non-negative polynomial of degree k and suppose it is polyharmonic of order p, i.e.*,

$$\Delta^p P_k = 0.$$

Then

$$k \leqslant 2(p-1).$$

Proof. The proof of this lemma is an easy consequence of Almansi expansion in combination with the orthogonality relation for spherical harmonics. First observe that we may assume P_k is homogeneous of degree k, since otherwise we consider $P_k(Rx)/R^k$ and let R tend to infinity. Next, P_k can be expressed as (see [ar])

$$P_k(x) = \sum_{j=0}^{p-1} |x|^{2j} h_j,$$

where h_j $(j = 0, ..., p-1)$ are harmonic polynomials of degree $k - 2j$. Now integrating the above expression over the unit sphere in $\mathbb{R}^N$ and using the non-negativity of P_k, we will have

$$0 < \int_{|x|=1} P_k = \sum_{j=0}^{p-1} \int_{|x|=1} h_j.$$

Since for h_j nonconstant $\int_{|x|=1} h_j = 0$, we conclude that $k - 2(p-1) \leqslant 0$, and thus the desired result. ∎

THEOREM 2.6. *Let Ω be an unbounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$, with $x^0 \notin \bar{\Omega}$, and $s \geqslant m+1$. Then the following holds.*

(i) *If $L_m = \Delta^p$ $(m = 2p)$, then*

$$\overline{\tilde{\mathfrak{F}}_s^+(\Omega, x^0)} = \tilde{\mathfrak{F}}^+(\Omega).$$

(ii) *If $m = 2$ and Conditions B holds or*

(iii) *if $m \geqslant 2$ and Condition A holds or*

(iv) *if $L_m = \Delta^p$ $(m = 2p)$ and Condition B holds, then*

$$\overline{\tilde{\mathfrak{F}}_s(\Omega, x^0)} = \tilde{\mathfrak{F}}(\Omega).$$

Proof. Since both sets in case (i) are convex cones, we need to prove that any non-negative functional on the smallest set is also non-negative on the larger one. Recall the beginning of the proof of Theorem 2.4. Then we need to show that if

$$U_s \geqslant 0 \qquad \text{on } \mathbb{R}^N, \tag{2.6}$$

and satisfies (2.3), then the same is true for U_{m+1}. Now (2.4), (2.6) and Theorem 2.1 imply

$$P_{s-1}(y) \leqslant U_{s-1}(y) \leqslant C\,|y|^{s-2} \log(2 + |y|) \qquad \forall y.$$

Dividing both sides of the above inequality by $|y|^{s-2}$ and letting $y \to \infty$, we will have $P_{s-1} \leqslant 0$ on $\mathbb{R}^N$. Since also $\Delta^p(P_{s-1}) = 0$, we may conclude by Lemma 2.1 that $P_{s-1} \equiv 0$ and thus $U_{s-1} \geqslant 0$. Repeating the same argument for $P_{s-2}, ..., P_{m+1}$, we will end up with $U_{m+1} \geqslant 0$, which is the desired result.

To prove (ii) and (iii) we start once again at (2.4). We thus need to prove $P_i \equiv 0$, for $m+1 \leqslant i \leqslant s-1$. Now let $k := \max\{i\colon m+1 \leqslant i \leqslant s-1,\ P_i \not\equiv 0\}$ and define

$$D_j := \{ y \in B(0,1);\ y^j + R_j y \notin \Omega \} \qquad (j = 1, 2, ...).$$

By Taylor's formula, for any α,

$$P_k^{(\alpha)}(y) = \frac{1}{R_j^{k-|\alpha|}} \left(P_k^{(\alpha)}(y^j + R_j\, y) - \sum_{\substack{|\beta| < k \\ \beta_i \geqslant \alpha_i}} P_k^{(\beta)}(y^j)\, \frac{(R_j\, y)^{\beta - \alpha}}{(\beta - \alpha)!} \right). \tag{2.7}$$

Since U_k and P_k coincide in $\mathbb{R}^N \backslash \Omega$, they must be identical in the open set $\mathbb{R}^N \backslash \bar{\Omega}$ and consequently, for all β (and not only for $|\beta| \leqslant m-1$) $U_k^{(\beta)} \equiv P_k^{(\beta)}$ in $B(y^j, \varepsilon)$ $(\subset \mathbb{R}^N \backslash \bar{\Omega})$. In particular $U_k^{(\beta)}(y^j) = P_k^{(\beta)}(y^j)$, for all β.

Using this and that $U_k^{(\alpha)}(y) = P_k^{(\alpha)}(y)$ in $\mathbb{R}^N \backslash \Omega$, for $|\alpha| \leqslant m-1$, we reduce (2.7) to

$$P_k^{(\alpha)}(y) = \frac{1}{R_j^{k-|\alpha|}} \left(U_k^{(\alpha)}(y^j + R_j y) - \sum_{\substack{|\beta| < k \\ \beta_i \geqslant \alpha_i}} U_k^{(\beta)}(y^j) \frac{(R_j y)^{\beta - \alpha}}{(\beta - \alpha)!} \right) \qquad \forall y \in D_j$$

and for all $|\alpha| \leqslant m-1$. Next we have the following assertion, which we prove later.

Assertion. For U_k, $\tilde{K}_k$, and $|\alpha| \leqslant m-1$ there holds

$$R_j^{(\alpha)} \left(U_k^{(\alpha)}(y^j + R_j y) - \sum_{\substack{|\beta| < k \\ \beta_i \geqslant \alpha_i}} U_k^{(\beta)}(y^j) \frac{(R_j y)^{\beta - \alpha}}{(\beta - \alpha)!} \right)$$

$$= \partial_y^{(\alpha)} \int_\Omega \tilde{K}_k(x, y^j + R_j y, y^j)\, g(x)\, dx.$$

By Assertion and the expression for $P_k^{(\alpha)}$, above, we have

$$P_k^{(\alpha)}(y) = \frac{1}{R_j^k} \partial_y^{(\alpha)} \int_\Omega \tilde{K}_k(x, y^j + R_j y, y^j)\, g(x)\, dx \qquad \forall y \in D_j, \quad |\alpha| \leqslant m-1.$$

Hence

$$\begin{aligned} |P_k^{(\alpha)}(y)| &\leqslant \frac{1}{R_j^{k-|\alpha|}} \int_\Omega |\partial_y^{(\alpha)} \tilde{K}_k(x, y^j + R_j y, y^j)|\, |g(x)|\, dx \\ &\leqslant \frac{C}{R_j^{k-|\alpha|}} \int_{\mathbb{R}^N \backslash B(y^j, \varepsilon)} |\partial_y^{(\alpha)} \tilde{K}_k(x, y^j + R_j y, y^j)|\, dx \\ &\leqslant \frac{C}{R_j^{k-|\alpha|}} \int_{\mathbb{R}^N \backslash B(0, \varepsilon)} |\partial_y^{(\alpha)} \tilde{K}_k(x, R_j y, 0)|\, dx \\ &\leqslant \frac{C}{R_j^{k-|\alpha|}} (R_j + 1)^{k-1-|\alpha|} \log(R_j + 2) \qquad \forall |\alpha| \leqslant m-1, \quad y \in D_j, \end{aligned}$$

where in the last estimate we have used Theorem 2.1. Letting $R_j \to \infty$ we conclude that

$$P_k^{(\alpha)}(y) \equiv 0, \qquad \forall y \in D_0, \quad |\alpha| \leqslant m-1,$$

where $D_0 := \overline{\lim}\, D_j$ is the set of all limit points of sequences $\{x^j\}$, $x^j \in D_j$. Hence by the homogeneity of P_k we have for $|\alpha| \leqslant m-1$, $P_k^{(\alpha)} \equiv 0$ in $\Lambda(0, 1, D_0)$.

Now Condition B implies

$$\varepsilon \leqslant \frac{\operatorname{cap}(A(y^j, R_j, \mathbb{R}^N \backslash \Omega))}{\operatorname{cap}(B(0, R_j))} = \operatorname{cap}(A(0, 1, D_j)) \qquad \forall j,$$

while Condition A gives

$$\varepsilon \leqslant \frac{|A(y^j, R_j, \mathbb{R}^N \backslash \Omega)|}{|B(0, R_j)|} = |A(0, 1, D_j)| \qquad \forall j.$$

Next letting j tend to infinity it is not hard to verify that the above inequalities imply

$$\varepsilon \leqslant \operatorname{cap}(A(0, 1, D_0)), \tag{2.8a}$$

$$\varepsilon \leqslant |A(0, 1, D_0)|. \tag{2.8b}$$

Indeed, for the volume case we have

$$|\overline{\lim}\, D_j| \geqslant \overline{\lim}\, |D_j|,$$

and a similar conclusion holds for the capacity function; see e.g. [ks1, (2.10)].

Now, for all $|\alpha| \leqslant m-1$, in case (ii) we will have $P_k^{(\alpha)} = 0$ in $A(0, 1, D_0)$, which according to (2.8a) has positive capacity. This is in contradiction with a theorem of L. Robbiano and J. Salazar [rs], unless $P_k \equiv 0$ in $\mathbb{R}^N$. In case (iii) we end up with a polynomial in $\mathbb{R}^N$ whose zero set has positive volume (by (2.8b)). This is indeed a contradiction, unless $P_k \equiv 0$.

As to the case (iv), we have

$$\Delta^p P_k = 0 \quad \text{in } \mathbb{R}^N \qquad \text{and} \qquad P_k^{(\alpha)} = 0 \quad \text{in } D_0,$$

for $|\alpha| \leqslant 2p-1$. Hence the polynomial $Q = \Delta^{p-1} P_k$ satisfies

$$\Delta Q = 0 \qquad Q^{(\alpha)} = 0 \quad \text{in } D_0,$$

where $|\alpha| \leqslant 1$. Now, by the assumption, (2.8b) is satisfied and thus the result in [rs] can be applied to deduce $Q = \Delta^{p-1} P_k \equiv 0$ in $\mathbb{R}^N$. Repeating this argument with $Q_j = \Delta^{p-j} P_k$, for $j = 2, 3, ..., p-1$, we will have the desired result. ∎

Proof of Assertion. Recall the definitions for $\tilde{K}_k$ and U_k, and notice that

$$\partial_y^{(\alpha)} \tilde{K}_k(x, y, z)|_{y=z} = 0 \qquad \text{and} \qquad \partial_y^{(\alpha)} U_k(z, z) = 0, \qquad \text{for} \quad z \in \mathbb{R}^N$$

and for all $|\alpha| < k$, provided $x \neq z$. To prove the assertion we fix y^j and define new functions

$$I_1(y) = I_1(y, y^j) = U_k^{(\alpha)}(y^j + R_j\, y, x^0) - \sum_{\substack{|\beta| < k \\ \beta_i \geqslant \alpha_i}} U_k^{(\beta)}(y^j)\, \frac{(R_j\, y)^{\beta - \alpha}}{(\beta - \alpha)!}$$

and

$$I_2(y) = I_2(y, y^j) = (\partial_y^{(\alpha)} U_k)(y^j + R_j\, y, y^j)$$

Now we claim

$$I_1(y) \equiv I_2(y),$$

which gives the desired result.

To prove this we observe that $I_1 - I_2$ solves $L_m(I_1 - I_2) = 0$ in $\mathbb{R}^N$ and hence is analytic in the entire space. It thus suffices to show that all derivatives of I_1 and I_2 coincide at $y = 0$, say. We remark that one may prove the equivalence of these functions by more elementary but tedious calculus.

Now for I_1 we obviously have

$$\partial^{(\sigma)} I_1(0) = R_j^{(\sigma)}\, U^{(\alpha + \sigma)}(y^j, x^0) - R_j^{(\sigma)}\, U^{(\alpha + \sigma)}(y^j, x^0) = 0,$$

for $|\sigma| < k - |\alpha|$, and for $|\sigma| \geqslant k - |\alpha|$

$$\partial^{(\sigma)} I_1(0) = \int_\Omega (-1)^{(\alpha + \sigma)} K^{(\alpha + \sigma)}(x - y^j)\, g(x)\, dx,$$

where the integral is bounded due to the fact that $B(y^j, \varepsilon) \cap \Omega = \varnothing$.

Similarly one obtains that $\partial^{(\sigma)} I_2(0) = U^{(\alpha + \sigma)}(y^j, y^j) = 0$ for $|\sigma| < k - |\alpha|$, and for $|\sigma| \geqslant k - |\alpha|$

$$\partial^{(\sigma)} I_2(0) = \int_\Omega (-1)^{(\alpha + \sigma)}\, K^{(\alpha + \sigma)}(x - y^j)\, g(x)\, dx. \quad \blacksquare$$

Remark. As the above proof shows, we can extract a stronger result in cases (ii)–(iv) than the one indicated in the theorem. Indeed, in these cases, the approximation problem is reduced to the problem of finding (estimating) the largest possible number of different hyper-surfaces of dimension $\leqslant (N-2)$, on which a nonzero polynomial solution to $L_m P = 0$ may vanish together with its $(m-1)$-derivatives.

Also, the restriction to Δ^p, in case (i), depends on the lack of a similar result as that in Lemma 2.1 for other operators L_m. However, we boldly conjecture the following.

Conjecture 1. Let P_k be a nonnegative polynomial of degree k and satisfy $L_m P_k = 0$, where L_m is uniformly elliptic as in Section 2. Then we conjecture that there is a constant $C(m, N)$ such that $k \leqslant C(m, N)$. We also suggest that $C(m, N) \leqslant N(m-2)$.

We remark that for $L_m = \partial_1^m + \cdots + \partial_N^m$ the polynomial $P(x) = (x_1 \cdots x_N)^{m-2}$ has the property that $L_m P = 0$ and $P \geqslant 0$. Moreover P is of degree $N(m-2)$. Wishful thinking suggests that this should be the only case with the exact growth $N(m-2)$.

EXAMPLE 2.3 ($N=2$). Let $N=2$ and $L_m = \Delta^p$, where $m=2p$. For a given Ω, we define the p-Schwarz potential of $u_p(z)$ to be the function (locally unique near $\partial\Omega$) which is polyharmonic near $\partial\Omega$ (at least in an interior neighborhood), and $(\partial^\alpha/\partial x^\alpha)(|x|^{2p} - u_p) = 0$, on $\partial\Omega$ for all $|\alpha| \leqslant 2p-1$.

Now let S_p be the p-Schwarz function defined in Example 1.2 and recall Example 1.3, according to which for any constant k there is an unbounded domain D (which depends on k) with a bounded complement, an analytic boundary, and a p-Schwarz function of growth $|z|^{p+k}$ near the infinity point. Since ∂D is analytic, it admits, by Cauchy–Kowalewski theorem, a p-Schwarz potential u_p. Now it is not hard to realize (by uniqueness) that $p!\, S_p = \partial^p u_p / \partial z^p$. Since S_p is polyanalytic in D (Example 1.3), it follows that u_p is polyharmonic in D. Moreover u_p behaves like $|x|^{2p+k}$ near infinity. Now let $k = s+1-2p$. Then we can argue as in Example 1.3 to conclude that for any Ω with bounded complement and $x^0 \notin \overline{\Omega}$

$$\overline{\tilde{\mathfrak{F}}_{s+1}(\Omega, x^0)} \neq \overline{\tilde{\mathfrak{F}}_s(\Omega, x^0)} \qquad \forall s \geqslant m+1.$$

EXAMPLE 2.4. ($N=3$) Let $\Omega \subset \mathbb{R}^3$ be such that $\mathbb{R}^3 \backslash \Omega$ is contained in a cylinder with $x^0 \notin \Omega$, and suppose $m=2$ and $L_m = \Delta^p$. Then we claim

$$\overline{\tilde{\mathfrak{F}}_{s+1}(\Omega, x^0)} \neq \overline{\tilde{\mathfrak{F}}_s(\Omega, x^0)}, \qquad s \geqslant m+1. \tag{2.9}$$

Let D and u_p be as in Example 2.3 with $k = s+1-2p$, and consider a cylindrical domain D' in $\mathbb{R}^3$ with base $D \subset \mathbb{R}^2$, i.e. $D' = D \times \mathbb{R}$. By translation and scaling we may assume $D' \subset \Omega$; cf. Example 1.3. Now define a functional T on $\mathscr{L}^1(\Omega)$ by

$$T(f) = \int_\Omega f(x)\, \chi_{D'}\, dx = \int_{D'} f(x)\, dx, \qquad \forall f \in \mathscr{L}^1(\Omega),$$

and let $u(x)$ be defined as

$$u(x) = \frac{(x_1^2 + x_2^2)^p - u_p(x_1, x_2)}{\text{const.}} \qquad \forall x \in D', \qquad u(x) = 0 \quad \text{in } \mathbb{R}^3 \backslash D',$$

where

$$\text{const.} = 2^p p!(N + 2(p-1))(N + 2(p-2)) \cdots (N+2)\, N.$$

Then, in $\mathbb{R}^3$, u solves the overdetermined problem

$$\begin{aligned} &2\Delta^p u = 1 && \text{in } D' \\ &\frac{\partial^\alpha u}{\partial x^\alpha} = 0 && \text{on } \partial D', \quad |\alpha| \leqslant 2p - 1 \\ &\sup_{|x| \leqslant R} |u(x)| \geqslant CR^s && \text{large } R, \qquad x \in D' \quad (C > 0). \end{aligned} \tag{2.10}$$

Using Green's identity and the fact that $D' \subset \Omega$ we see that T vanishes on $\tilde{\mathfrak{F}}_{s+1}(\Omega)$.

Now suppose (2.9) does not hold, i.e. T also vanishes on $\tilde{\mathfrak{F}}_s$. Then this implies that U_s solves the overdetermined system (2.10), and thus is identical with u in $\mathbb{R}^3$; due to unique continuation. Since $U_s = U_{m+1} + \sum_{i=m+1}^{s-1} P_i$ we conclude by Theorem 2.1 that $|u(x)| = |U_s(x)| \leqslant C(|x|^{s-1} + 1) \log(|x| + 2)$, contradicting (2.10). This proves (2.9).

3. REMARKS AND FURTHER HORIZONS

The assumptions of Theorem 2.6 seems superfluous, as the reader may already have noticed. Indeed, let $P(x)$ be a polynomial of degree $(k \geqslant m)$ that satisfies the elliptic equation $L_m P = 0$. Define the critical zero set of P by

$$\mathscr{N}_P = \{x \colon D^\alpha P(x) = 0, |\alpha| < m\}.$$

Obviously if $\mathscr{N}_P$ is large, then $P \equiv 0$. The question is how large this set can be. Our measurements for largeness (Conditions A and B) is of course far from being optimal or sharp. It is the lack of a general uniqueness theory for such polynomials that forces us to use these conditions.

However, due to results of Q. Han [h] the set $\mathscr{N}_P$ consists of a finite number of C^1 manifolds of Hausdorff dimension $\leqslant (N-2)$. Actually Q. Han proves this for general solutions of $L_m u = 0$.

In particular in $\mathbb{R}^3$, the critical zero set for any homogeneous harmonic polynomial consists of the union of a finite number of lines. The number of these lines should of course depend on the degree of the polynomial. It is quite tantalizing to find the exact number of these lines in terms of the degree of the polynomial. We will illustrate this by the following example.

EXAMPLE 3.1. In $\mathbb{R}^3$, let $P \not\equiv 0$ be a homogeneous harmonic polynomial of degree 3, and set $\mathcal{N}(P) = \{x\colon P(x) = |\nabla P(x)| = 0\}$, which we call the critical zero set of P. By the previous discussions we may assume that either $\mathcal{N}(P) = \{0\}$, or there are lines l_j through the origin such that $\mathcal{N}(P) = \bigcup_{j=1}^{k} l_j$, for some $k \geqslant 1$.

Suppose the latter holds. Then we claim $k \leqslant 3$. To cope with this problem we may use rotation invariance and assume that l_1 is the x_1-axis and $l_2 \subset \{x_3 = 0\}$. Set now

$$P = \sum_{j=0}^{3} x_1^{3-j} h_j,$$

where h_j is a homogeneous polynomial of degree j and independent of x_1. Since $P = 0$ on the x_1-axis and h_0 is constant, we must have $h_0 = 0$. Next, on the x_1-axis

$$\begin{aligned}\nabla P &= (2x_1 h_1 + h_2,\, x_1^2\, \partial_2 h_1 + x_1\, \partial_2 h_2 + \partial_2 h_3,\, x_1^2\, \partial_3 h_1 + x_1\, \partial_3 h_2 + \partial_3 h_3)\\ &= (0,\, x_1^2\, \partial_2 h_1,\, x_1^2\, \partial_3 h_1) = (0, 0, 0),\end{aligned}$$

which gives that h_1 is constant and hence zero. It thus follows that

$$P = x_1 h_2 + h_3,$$

where h_j $(j = 2, 3)$ is a harmonic homogeneous polynomial of degree j and independent of x_1. Since $\partial_1 P(x_1, x_2, 0) = h_2(x_2, 0) = 0$ on l_2 and h_2 is harmonic, we must have (after if necessary dividing by a constant) $P = x_1 x_2 x_3 + h_3$. Now $P = 0$ on l_2 gives $0 = P = h_3(x_2, 0)$ on $l_2 \subset \{x_3 = 0\}$. Hence $h_3 = x_3 h$ for some polynomial h of degree 2, and independent of x_1. Using that P is harmonic we have

$$0 = \Delta P = x_3\, \Delta h + 2\partial_3 h \qquad \text{in } \mathbb{R}^3$$

and in particular on $\{x_3 = 0\}$, which gives $\partial_3 h = 0$ on $\{x_3 = 0\}$, i.e., $\partial_3 h = a x_3$ for some constant a, hence $P = x_1 x_2 x_3 + b(x_3^3 - 3x_3 x_2^2)$ for some constant b. Now, by elementary calculus, $\partial_3 P = 0$ on $l_2 \subset \{x_3 = 0\}$ implies that $\mathcal{N}(P) \cap \{x_3 = 0\} = \{x_1 = 3a x_2, x_3 = 0\} \cup l_1$, i.e., $l_2 = \{x_1 = 3a x_2, x_3 = 0\}$. Hence if $k \geqslant 3$ then l_j $(j \geqslant 3)$ does not lie in the plane $\{x_3 = 0\}$. Since in the rotation the choice of l_2 was arbitrary we may conclude that non of the lines $l_j \neq l_1$ lie in the same plane as that generated by l_1 and l_i

for $i \neq j$. To sum up we have that the harmonic polynomial $h_2 = \partial_1 P$ (in x_2, x_3 variables) vanishes on the projection of the lines l_j ($j \geqslant 2$) on the $x_2 x_3$-plane. Since h_2 has degree 2, it can not vanish on more than two lines. Hence $k \leqslant 3$.

The above example and the discussion preceding it, can be used to prove Propositions 3.1–3.2 below. First we need a definition.

Condition $C(n)$. A given domain Ω is said to satisfy Condition $C(n)$ (n being a positive integer) if there is a sequence of points $\{y^j\}$ in $\mathbb{R}^N \backslash \Omega$, positive numbers $R_j \to \infty$, and $\varepsilon > 0$ such that $B(y^j, \varepsilon) \cap \Omega = \varnothing$ and the truncated cone $\Lambda(y^j, R_j, \mathbb{R}^N \backslash \Omega)$ contains at least $n+1$ line segments emanating from y^j and with a length proportional to R_j and such that the angle between any pair of such segments is less than $\pi - \varepsilon$, i.e., the limit set D_0 in the proof of Theorem 2.5 contains at least $n+1$ lines through the origin.

PROPOSITION 3.1. *Let Ω be an unbounded domain in $\mathbb{R}^3$, with $x^0 \notin \Omega$, and suppose $L_m = \Delta^p$ ($m = 2p$). Then for each $s \geqslant m+1$ there exists a (large) $n = n(s) > 0$ such that if Condition $C(n)$ holds for Ω, then*

$$\overline{\tilde{\mathfrak{F}}_s(\Omega, x^0)} = \tilde{\mathfrak{F}}(\Omega).$$

PROPOSITION 3.2. *Let Ω be an unbounded domain in $\mathbb{R}^3$, with $x^0 \notin \bar{\Omega}$, and let $L_m = \Delta^p$ ($m = 2p$). Suppose also Condition $C(3)$ holds for Ω. Then*

$$\overline{\tilde{\mathfrak{F}}_4(\Omega, x^0)} = \tilde{\mathfrak{F}}(\Omega).$$

Proof of Propositions 3.1 *and* 3.2. First suppose $m = 2$ (i.e. $p = 1$). Then following the lines of the proof in Theorem 2.5, we need to show that the polynomial P_k, whose critical zero set is D_0, is identically zero. But then Example 3.1 (for $k = 3$) and the discussion preceding it (for the general case) gives that $P_k \equiv 0$.

Now for $m = 2p$ and $p > 1$ we argue as in the proof of part (iv) in Theorem 2.5. The proof is completed. ▮

To find the exact value for $C(n)$, remains an open and tantalizing problem.

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